PERGAMON

International Journal of Solids and Structures 37 (2000) 6155–6171

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

www.elsevier.com/locate/ijsolstr

Integral representation and Green's functions for 3D time-dependent thermo-piezoelectricity

L.Z. Jiang [1]

School of Aeronautics and Astronautics, Purdue University, West Lafayette, IN 47906, USA

Received 24 June 1999; in revised form 22 August 1999

Abstract

A boundary integral method is developed for 3D time-dependent quasistatic thermo-piezoelectricity. This time domain formulation involves only surface quantities. As a result, volume discretization is completely avoided. Green's functions are derived by resorting to Radon transform and presented in integral form. Reductions to the case of isotropic dielectric solid have been worked out in detail and its corresponding solutions expressed in explicit form are consistent with the existing solutions. © 2000 Published by Elsevier Science Ltd.

Keywords: Boundary integral equation; Thermal stress; Thermo-piezoelectricity; Green's solution

1. Introduction

Piezoelectric materials exhibit coupling behavior between elastic and electric fields and are inherently anisotropic. As a result, they have been widely used as electromechanical devices. Due to the possibility in sensing or actuating the deformation of a piezoelectric body by controlling electric field, piezoelectric materials also have potential applications in smart structures. However, these electromechanical devices are often placed in a hostile environment and must be designed to withstand thermal transients which may cause excessive shock, fatigue and rupture (Jiang and Sun, 1999). Hence, the thermal effects on the performance of piezoelectric sensors and actuators are of great interest. A review of existing literature reveals that studies on piezoelectric materials attracted numerous researchers in the last few years (Sosa, 1991; Pak, 1992; Suo et al., 1992; Lee and Jiang, 1996) and studies on thermo-piezoelectric materials have received some attention as well. Tiersten (1971) derived non-linear equation of thermo-electro-elasticity, and Massalas et al. (1994) recently extended Tiersten's theory by including the "second

E-mail address: longzhi.jiang@med.ge.com (L.Z. Jiang).
[1] Present address: General Electric Company, P.O. Box 100539, Florence, SC 29501.

PII: S0020-7683(99)00252-8

sound". Ashisa et al. (1983) considered the piezothermoelastic behavior of an axially heated thin piezoelectric plate by using the potential function method. Dunn (1993) derived effective thermal expansion and pyroelectric coefficients of two-phase coupled electroelastic composite materials. Tzou (1993) presented the finite element formulation for a steady state piezothermoelastic solid. Jonnalagadda et al. (1994) discussed the response of a thin plate constructed of piezothermoelastic layers by choosing plate theory involving shear deformation.

However, all of the studies on thermo-piezoelectricity cannot be easily extended to analysis of thermal fracture behavior and micromechanics of such brittle materials as piezoceramics even though it is important to understand the behavior in order to efficiently design smart structures (Jiang and Sun, 1999). This motivates the author to study Green's solutions for these materials and further develop a boundary element method in exploiting fracture and micromechanics. With Green's solutions, the boundary integral–boundary element method can be formulated and implemented and the aforementioned analyses can be constructed as in the case of pure elasticity (Mura, 1984).

Before deriving the formulation for thermo-electro-elasticity, a brief review of the existing literature is appropriate. Some of the relevant work has appeared within the context of dynamical ansiotropic elasticity and piezoelectricity. Bacon et al. (1979) investigated time-independent Green's function of anisotropic media thoroughly. Wang and Achenbach (1994) and Wang and Achenbach (1995) derived time-harmonic Green's solution of anisotropic solid. Analytic expression of Green's solution for general anisotropic piezoelectric media was derived by Deeg (1980), and corresponding numerical techniques were explained in detail. Chen (1993) and Chen and Lin (1993) rederived the Green's function by using the Fourier transform and discussed the possibility of using their solution in numerical methods. For man-made materials that behave as transversely isotropic media, Lee and Jiang (1994a) obtained closed form solution by means of retaining the finite part of a divergent integral, and Dunn (1994a, 1994b) simplified Deeg's general solutions and obtained closed form solutions. Norris (1994) obtained Green's solution for harmonic dynamical piezoelectricity. Khutoryansky and Sosa (1995) presented their dynamical Green's function for a general anisotropic material, which is represented as an integral over the slowness surface on an arbitrary piezoelectric body. Ding et al. (1997) derived fundamental solutions for two-phase transversely isotropic piezoelectric media.

However, a general time-dependent thermo-electro-elastic solid poses formidable mathematical complexities. It appears that relatively little work has been done regarding the studies of Green's functions except that Lee and Jiang (1994b) considered Green's function for steady state transversely isotropic thermo-electro-elastic solid.

In this study, a three-dimensional boundary integral formulation is developed for time-dependent thermo-piezoelectric solid. The techniques employed in the derivation for Green's function are elucidated and Green's solutions are presented in integral form. The method operates directly in the time domain and most importantly, requires no volume discretization. Thus, transient thermo-piezoelectric analysis may be accomplished from a model consisting exclusively of surface elements. Not only does this considerably reduce the manpower requirements for modeling, but with the Green's functions, the unknown quantities near the surface can be captured much more readily than with domain based methods.

2. Governing equations

The differential equations governing the behavior of a thermo-piezoelectric solid under quasistatic conditions where the initial force is negligible can be written as (Nowacki, 1979; Deeg, 1980)

$$\sigma_{ij,j} + F_i = 0 \tag{1a}$$

$$D_{i,i} - \Phi_e = 0 \tag{1b}$$

where σ_{ij} is the stress tensor, D_i is the electric displacement vector, F_i is the body force density vector, and Φ_e is the body charge density. The repeated suffix obeys the Einstein summation convention. In terms of entropy balance, the heat conduction equation is

$$q_{i,i} - T\dot{s} + \Phi_t = 0 \tag{1c}$$

where the over dot represents differentiation with respect to time, T is the instantaneous temperature, q_i is the heat flux, s is the entropy density, and Φ_t is the heat source density.

In what follows, assume $\frac{T_0 - T}{T} \ll 1$, we arrive at the linear heat conduction equation as

$$q_{i,i} - T_0\dot{s} + \Phi_t = 0 \tag{1d}$$

where T_0 is the stress-free reference temperature. The constitutive equations are (Nowacki, 1979)

$$\sigma_{ij} = c_{ijkl}s_{kl} - e_{kij}E_k - \lambda_{ij}\theta, \tag{2a}$$

$$D_i = e_{ikl}s_{kl} + \varepsilon_{ik}E_k + \pi_i\theta, \tag{2b}$$

$$s = \lambda_{kl}s_{kl} + \pi_k E_k + \frac{\rho c_\varepsilon \theta}{T_0} \tag{2c}$$

and Fourier law for anisotropic body becomes

$$q_i = k_{ij}Q_j \tag{2d}$$

where s_{ij} is the strain tensor, E_k is the electric field vector, θ is the temperature change from T_0, i.e. $\theta = T - T_0$, Q_i is the temperature gradient, ρ is the mass density, c_ε is the specific heat, c_{ijkl} is the elastic constant tensor measured at constant field, e_{kij} is the piezoelectric constant tensor, ε_{ik} is the dielectric constant tensor measured at constant strain, λ_{ij} is the thermoelastic constant tensor, π_i is the pyroelectric constant vector and k_{ij} is the thermal conductivity tensor. These constants obey the following symmetric relations (Nye, 1957):

$$c_{ijkl} = c_{ijlk} = c_{jikl} = c_{klij} = c_{jilk}, \tag{3a}$$

$$e_{kij} = e_{kji} \tag{3b}$$

$$\varepsilon_{ik} = \varepsilon_{ki}, \quad k_{ij} = k_{ji}, \quad \lambda_{ij} = \lambda_{ji}. \tag{3c}$$

Gradient equations:

$$s_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \tag{4a}$$

$$E_i = -\phi_{,i} \tag{4b}$$

$$Q_i = \theta_{,i} \tag{4c}$$

where u_i and ϕ are the displacement in the ith direction and electric potential, respectively. For a well-posed problem, appropriate boundary conditions and initial conditions have to be specified.

It is noted that the theory portrayed by Eqs. (1a)–(4c) addresses the full coupling among thermal, electric and elastic fields. In general, these equations must be solved simultaneously. However, from an engineering viewpoint, it has been determined by a number of researchers in thermoelasticity (e.g. Boley and Weiner, 1960) that the term involving displacement in Eq. (2c) is negligible. In particular, for some engineering problems, the general theory may be further simplified by removing the time-dependent nature of the problem. As a consequence, it is assumed that the loads are applied slowly so that all the resulting diffusive processes have been finished. In other words, the body is presumed to have reached the steady state. As a result, the heat conduction equation becomes

$$q_{i,\,i} + \Phi_t = 0. \tag{5}$$

In this case, Eqs. (2d), (4c) and (5) can be first solved independently for the temperature field. Subsequently, displacements and electric potential are determined from rest of the equations with the known temperature field (Lee and Jiang, 1995). However, in this study, boundary integral formulation and its Green's functions for the coupled thermo-piezoelectricity are discussed.

3. Boundary integral formulation

Applying the Laplace transform (Courant and Hilbert, 1962) with respect to time to Eqs. (1a), (1b) and (1d) leads to

$$\bar{\sigma}_{ij,\,j} + \bar{F}_i = 0, \tag{6a}$$

$$\bar{D}_{i,\,i} - \bar{\Phi}_e = 0 \tag{6b}$$

$$\bar{q}_{i,\,i} - T_0\beta\bar{s} + \bar{\Phi}_t = 0 \tag{6c}$$

where β is the Laplace transform variable and $s(0)$ is assumed to be zero without loss of generality. The constitutive relations in the transform domain yield

$$\bar{\sigma}_{ij} = c_{ijkl}\bar{s}_{kl} - e_{kij}\bar{E}_k - \lambda_{ij}\bar{\theta}, \tag{7a}$$

$$\bar{D}_i = e_{ikl}\bar{s}_{kl} + \varepsilon_{ik}\bar{E}_k + \pi_i\bar{\theta}, \tag{7b}$$

$$\bar{s} = \lambda_{kl}\bar{s}_{kl} + \pi_k\bar{E}_k + \frac{\rho c_\varepsilon\bar{\theta}}{T_0} \tag{7c}$$

$$\bar{q}_i = k_{ij}\bar{\theta}_{,j}. \tag{7d}$$

In the transform domain, the boundary integral equation may be derived from the weighted residual statement as follows (Lee and Jiang, 1994a, 1994b):

$$\int_\Omega \left(\bar{\sigma}_{ij,\,j} + \bar{F}_i\right)\bar{u}_i^*\,\mathrm{d}\Omega = 0 \tag{8}$$

where $\bar{u}_i^*$ is the displacement in transform domain corresponding to the weighting field. By using integration by parts, divergence theorem and constitutive equations, we obtain the following expression:

$$\begin{aligned}&\int_\Omega \bar{\sigma}^*_{kl,\,l}\bar{u}_k\,\mathrm{d}\Omega + \int_\Omega \bar{D}^*_{m,\,m}\bar{\phi}\,\mathrm{d}\Omega + \int_\Gamma n_j\bar{\sigma}_{ij}\bar{u}_i^*\,\mathrm{d}\Gamma - \int_\Gamma n_j\bar{\sigma}^*_{ij}\bar{u}_i\,\mathrm{d}\Gamma + \int_\Omega \bar{F}_i\bar{u}_i^*\,\mathrm{d}\Omega - \int_\Omega \bar{\Phi}_e\bar{\phi}^*\,\mathrm{d}\Omega \\ &+ \int_\Gamma n_m\bar{D}_m\bar{\phi}^*\,\mathrm{d}\Gamma - \int_\Gamma n_m\bar{D}^*_m\bar{\phi}\,\mathrm{d}\Gamma + \int_\Omega\left(\pi_m\bar{E}^*_m\bar{\theta} - \pi_m\bar{E}_m\bar{\theta}^*\right)\mathrm{d}\Omega + \int_\Omega\left(\lambda_{ij}\bar{s}^*_{ij}\bar{\theta} - \lambda_{ij}\bar{s}_{ij}\bar{\theta}^*\right)\mathrm{d}\Omega \\ &= 0. \qquad (9)\end{aligned}$$

In order to eliminate the last two volume integrals as required in boundary integral method, we apply the weighting field and actual field to the linear heat conduction equation and obtain

$$k_{ij}\bar{\theta}^*_{,ij} - T_0\beta\bar{s}^* + \bar{\Phi}^*_t = 0 \qquad (10a)$$

$$k_{ij}\bar{\theta}_{,ij} - T_0\beta\bar{s} + \bar{\Phi}_t = 0. \qquad (10b)$$

After somewhat lengthy manipulation, such as subtracting Eq. (10a) multiplied by θ from Eq. (10b) multiplied by θ^*, integrating by parts, using Eq. (7c) and taking the integration over the domain occupied by the piezoelectric solid, we arrive at

$$\begin{aligned}&\int_\Omega\left(\pi_m\bar{E}^*_m\bar{\theta} - \pi_m\bar{E}_m\bar{\theta}^*\right)\mathrm{d}\Omega + \int_\Omega\left(\lambda_{ij}\bar{s}^*_{ij}\bar{\theta} - \lambda_{ij}\bar{s}_{ij}\bar{\theta}^*\right)\mathrm{d}\Omega \\ &= \frac{1}{\beta T_0}\left[\int_\Omega\left(\bar{\Phi}^*_t\bar{\theta} - \bar{\Phi}_t\bar{\theta}^*\right)\mathrm{d}\Omega - \int_\Gamma\left(\bar{q}^*\bar{\theta} - \bar{q}\theta^*\right)\mathrm{d}\Gamma\right]. \qquad (11)\end{aligned}$$

Recall the property of the inverse Laplace transform, i.e.

$$L^{-1}\left(\beta\bar{f}\right) = \dot{f} \quad \text{where} \quad L^{-1}\left(\bar{f}\right) = f. \qquad (12)$$

Note the convolution integral, namely,

$$a\times b = \int_0^t a(t-\tau)b(\tau)\,\mathrm{d}\tau = \int_0^t a(t)b(t-\tau)\,\mathrm{d}\tau. \qquad (13)$$

Substituting Eq. (11) into Eq. (9) and applying the inverse Laplace transform to Eq. (9), we arrive at

$$\begin{aligned}&\int_\Omega \dot{\sigma}^*_{kl,\,l}\times u_k\,\mathrm{d}\Omega + \int_\Omega \dot{D}^*_{m,\,m}\times\phi\,\mathrm{d}\Omega + \frac{1}{T_0}\int_\Omega \Phi^*_t\times\theta\,\mathrm{d}\Gamma + \int_\Omega f_i\times\dot{u}_i^*\,\mathrm{d}\Omega - \int_\Omega \Phi_e\times\dot{\phi}^*\,\mathrm{d}\Omega \\ &- \frac{1}{T_0}\int_\Omega \Phi_t\times\theta_i^*\,\mathrm{d}\Gamma - \int_\Gamma t_i\times\dot{u}_i^*\,\mathrm{d}\Gamma + \int_\Gamma \omega\times\dot{\phi}_i^*\,\mathrm{d}\Gamma - \frac{1}{T_0}\int_\Omega q\times\theta^*\,\mathrm{d}\Gamma \\ &- \int_\Gamma \dot{t}_i^*\times u_i\,\mathrm{d}\Gamma + \int_\Gamma \dot{\omega}^*\times\phi\,\mathrm{d}\Gamma - \frac{1}{T_0}\int_\Omega q^*\times\theta\,\mathrm{d}\Gamma = 0\end{aligned}$$

where $t_i = n_j\sigma_{ji}$, $q = -k_{ij}\theta_{,i}n_j$ and $\omega = -n_jD_j$. In order to make the first three volume integrals vanish, we consider the weighting field as an infinite piezoelectric body subjected to unit forces, electric charge and pulse heat source equal to the following three states separately.

$$F_i^* = \delta\left(\vec{\xi}, \vec{x}\right)H(t)\delta_{ij}e_j, \quad \Phi_e^* = 0, \quad \Phi_t^* = 0, \tag{15a}$$

$$F_4^* = 0, \quad \Phi_e^* = \delta\left(\vec{\xi}, \vec{x}\right)H(t), \quad \Phi_t^* = 0 \tag{15b}$$

$$F_5^* = 0, \quad \Phi_e^* = 0, \quad \Phi_t^* = \delta\left(\vec{\xi}, \vec{x}\right)\delta(t) \tag{15c}$$

where δ is the Dirac delta function and $H(t)$ is the Heaviside step function. It is noted that the heat source with respect to time t is subjected to the Dirac delta function while the other two loads are subjected to the step function. It is purely from the mathematical consideration and can be easily explained by noting that Φ_t^* in the integral does not involve time derivative while both $\dot{\sigma}_{kl,\,l}^*$ and $\dot{D}_{m,\,m}^*$ do. For the sake of brevity, let us assume that there are no body forces, volume charge or heat source in the actual boundary value problem. The boundary integral equation for coupled thermo-piezoelasticity then becomes

$$\begin{aligned} U_I\left(\vec{\xi}\right) + \int_\Gamma \dot{t}_{Ij}^*\left(\vec{\xi}, \vec{x}\right) \times u_j(\vec{x})\,\mathrm{d}\Gamma - \int_\Gamma \dot{\omega}_I^*\left(\vec{\xi}, \vec{x}\right) \times \phi(\vec{x})\,\mathrm{d}\Gamma + \frac{1}{T_0}\int_\Gamma q_I^*\left(\vec{\xi}, \vec{x}\right) \times \theta(\vec{x})\,\mathrm{d}\Gamma \\ = \int_\Gamma \dot{u}_{Ij}^*\left(\vec{\xi}, \vec{x}\right) \times t_j(\vec{x})\,\mathrm{d}\Gamma - \int_\Gamma \dot{\phi}_I^*\left(\vec{\xi}, \vec{x}\right) \times \omega(\vec{x})\,\mathrm{d}\Gamma + \frac{1}{T_0}\int_\Gamma \theta_I^*\left(\vec{\xi}, \vec{x}\right) \times q(\vec{x})\,\mathrm{d}\Gamma \end{aligned} \tag{16}$$

where $U_I = u_i$ $(I = 1, 2, 3)$, $U_4 = -\phi$ and $U_5 = -\theta/T_0$. u_{ij}^* represents the displacement in jth direction at a field point $\vec{x}$ due to a point force in the ith direction at the source point $\vec{\xi}$ interior to Γ, u_{4i}^* denotes the ith displacement at x due to a point electric charge $\vec{\xi}$, u_{5i}^* stands for the ith displacement at $\vec{x}$ due to pulse heat source acting at time zero and at point $\vec{\xi}$, and so on. Eq. (16) can be viewed as a generalized Somigliana's identity for coupled time-dependent thermo-piezoelectricity, and, as such, is an exact statement for the interior displacements, electric potential and temperature. The process of obtaining Eq. (16) for a position on the boundary is not trivial due to the singular nature of the kernel functions. However, the singularity reflecting the smoothness of boundary can be obtained in explicit form for simple geometric configuration and, in general, to be determined by using rigid body motion as in the static anisotropic elasticity (Jiang and Lee, 1994). It should be noted that the integrals are Cauchy principal valued.

4. Green's solution for fully coupled thermo-piezoelectric media

It is customary, as in Eq. (16), that the quantities reflecting Green's function are asterisked in the boundary element realm. For simplicity, however, the asterisk will be omitted in what follows. After somewhat lengthy manipulation, such as substituting Eqs. (2a)–(2d) into Eqs. (1a)–(1d) and using Eqs. (3a)–(4c), we obtain a set of equations in terms of displacements, electric potential, temperature and entropy density as

$$\left(c_{ijkl} + \frac{T_0\lambda_{ij}\lambda_{kl}}{\rho c_\varepsilon}\right)u_{k,\,lj} + \left(e_{ijl} - \frac{T_0\lambda_{ij}\pi_l}{\rho c_\varepsilon}\right)\phi_{,lj} - \frac{T_0\lambda_{ij}}{\rho c_\varepsilon}s_{,j} + F_i\delta\left(\vec{\xi}, \vec{x}\right)H(t) = 0, \tag{17a}$$

$$\left(e_{lij}-\frac{T_0\lambda_{ij}\pi_l}{\rho c_\varepsilon}\right)u_{i,\,jl}-\left(\varepsilon_{ik}-\frac{T_0\pi_i\pi_k}{\rho c_\varepsilon}\right)\phi_{,ik}+\frac{T_0\pi_i}{\rho c_\varepsilon}s_{,i}-\Phi_e\delta\left(\vec{\xi},\vec{x}\right)H(t)=0 \tag{17b}$$

$$k_{ij}\theta_{,ij}-T_0\dot{s}+\Phi_t\delta\left(\vec{\xi},\vec{x}\right)\delta(t)=0. \tag{17c}$$

Due to its conciseness, the Radon transform is employed here for carrying out the derivation. A detail discussion of the Radon transform is given in Courant and Hilbert (1962) and Gelfand and Shilow (1964). Also, a number of its properties are presented in Bacon et al. (1979), Deans (1983), Deeg (1980) and Wang and Achenbach (1993). Using the notation presented there, the Radon transform of a function $f(\vec{x})$, which can be displacement, electric potential, temperature or entropy density, is defined as

$$\hat{f}=\hat{f}(\alpha,\vec{z})=\int_{\vec{z}\cdot\vec{x}=\alpha}f(\vec{x})\,\mathrm{d}s(\vec{x}) \tag{18a}$$

where $\vec{z}$ and α are the transform space variables and $\hat{f}$ is the integral of f over the plane defined by $\vec{z}\cdot\vec{x}=\alpha$. For simplicity, $\vec{z}$ is chosen to be a unit vector.

The inverse Radon transform for a three-dimensional domain is given by

$$f(\vec{x})=-\frac{1}{8\pi^2}\int_{|\vec{z}|=1}\frac{\partial^2\hat{f}(\alpha,\vec{z})}{\partial\alpha^2}|_{\vec{z}\cdot\vec{x}=\alpha}\,\mathrm{d}s(\vec{z}). \tag{18b}$$

The integral is carried out over the surface of the unit sphere $|\vec{z}|=1$. Applying the Radon transform to both sides of Eqs. (17a) and (17b) leads to

$$\left(c_{ijkl}+\frac{T_0\lambda_{ij}\lambda_{kl}}{\rho c_\varepsilon}\right)z_jz_l\hat{u}_k''+\left(e_{ijl}-\frac{T_0\lambda_{ij}\pi_l}{\rho c_\varepsilon}\right)z_jz_l\hat{\phi}''-\frac{T_0\lambda_{ij}}{\rho c_\varepsilon}z_j\hat{s}'+F_i\delta(\alpha)H(t)=0,$$

$$\left(e_{lij}-\frac{T_0\lambda_{ij}\pi_l}{\rho c_\varepsilon}\right)z_lz_j\hat{u}_i''-\left(\varepsilon_{ik}-\frac{T_0\pi_i\pi_k}{\rho c_\varepsilon}\right)z_iz_k\hat{\phi}''-\frac{T_0\pi_i}{\rho c_\varepsilon}z_i\hat{s}'-\Phi_e\delta(\alpha)H(t)=0. \tag{19}$$

Solving the above two equations simultaneously for $\hat{u}_k''$ and $\hat{\phi}''$ in terms of $\hat{s}'$ yields

$$\hat{u}_k''=-c_k\hat{s}'-h_{ki}F_i\delta(\alpha)H(t)-d_k\Phi_e\delta(\alpha)H(t) \tag{20a}$$

$$\hat{\phi}''=-e\hat{s}'-p_iF_i\delta(\alpha)H(t)-q\Phi_e\delta(\alpha)H(t) \tag{20b}$$

where

$$b_{ik}=\left(c_{imkn}+\frac{T_0}{\rho c_\varepsilon}\lambda_{im}\lambda_{kn}\right)z_mz_n+\frac{1}{a}\left[e_{ijl}e_{mnk}-\left(\frac{T_0}{\rho c_\varepsilon}\right)^2\lambda_{ij}\lambda_{kn}\pi_l\pi_m\right]z_jz_nz_lz_m, \tag{21a}$$

$$a=\left(\varepsilon_{ik}-\frac{T_0\pi_i\pi_k}{\rho c_\varepsilon}\right)z_iz_k, \tag{21b}$$

$$h_{ki}=(b_{ik})^{-1}, \tag{21c}$$

$$c_k = \left(-\frac{1}{a} e_{ijl}\pi_m z_j z_l z_m + \frac{T_0}{a\rho c_\varepsilon} \lambda_{ij}\pi_l \pi_m z_j z_l z_m - \frac{T_0 \lambda_{in}}{\rho c_\varepsilon} z_n \right) h_{ki}, \tag{21d}$$

$$d_i = -\frac{1}{a}\left(e_{ijl} z_j z_l - \frac{T_0}{\rho c_\varepsilon} \lambda_{kj}\pi_l z_j z_l \right) h_{ik}, \tag{21e}$$

$$e = \frac{1}{a}\left[\frac{T_0 \pi_i z_i}{\rho c_\varepsilon} + c_k \left(e_{ilk} + \frac{T_0}{\rho c_\varepsilon} \lambda_{kl}\pi_i \right) z_i z_l \right], \tag{21f}$$

$$p_i = \frac{1}{a}\left(e_{mlk} + \frac{T_0}{\rho c_\varepsilon} \lambda_{kl}\pi_m \right) z_m z_l h_{ki} \tag{21g}$$

$$q = \frac{1}{a}\left[1 + \left(e_{ijk} + \frac{T_0}{\rho c_\varepsilon} \lambda_{kj}\pi_i \right) z_i z_j d_k \right]. \tag{21h}$$

Applying the Radon transform to the constitutive relation (2c) and taking the derivative with respect to α twice, we arrive at

$$\hat{\theta}'' = \frac{T_0\left(\hat{s}'' - \lambda_{ij} z_j \hat{u}_i''' + \pi_i z_i \hat{\phi}''' \right)}{\rho c_\varepsilon}. \tag{22}$$

Substituting $\hat{u}_i''$ and $\hat{\phi}_i''$ into the above equation leads to the relation between $\hat{\theta}''$ and $\hat{s}''$ as

$$\hat{\theta}'' = g\hat{s}'' + v_i F_i \delta'(\alpha) H(t) + w\Phi_e \delta'(\alpha) H(t) \tag{23}$$

where

$$g = \frac{T_0}{\rho c_\varepsilon}(1 + \lambda_{mn} c_n z_m - \pi_m z_m e)$$

$$v_i = \frac{T_0}{\rho c_\varepsilon}(\lambda_{km} h_{ki} z_m - \pi_m z_m p_i)$$

$$w = \frac{T_0}{\rho c_\varepsilon}\left(\lambda_{kj} z_j d_k - \pi_m z_m q\right). \tag{24}$$

Applying Radon transform to Eq. (17c) leads to

$$k_{ij} z_i z_j \hat{\theta}'' - T_0 \dot{\hat{s}} + \Phi_t \delta(\alpha)\delta(t) = 0. \tag{25}$$

As a result, substituting Eq. (23) into Eq. (25), the propagation equation in terms of entropy density is expressed in the following form

$$\dot{\hat{s}} - A\hat{s}'' = \frac{\Phi_t}{T_0}\delta(\alpha)\delta(t) + B_i F_i \delta'(\alpha) H(t) + C\Phi_e \delta'(\alpha) H(t) \tag{26}$$

where

$$A = \frac{k_{ij} z_i z_j g}{T_0}$$

$$B_k = \frac{k_{ij} z_i z_j v_k}{T_0}$$

$$C = \frac{k_{ij} z_i z_j w}{T_0}. \tag{27}$$

It is observed in Eq. (26) that the full coupling among thermal, electric and elastic fields occurs. From the computational viewpoint, once $\hat{s}''$ is evaluated, temperature, displacement and electric potential will be obtained using Eqs. (20a), (20b) and (23), respectively. In what follows, the expressions of these quantities corresponding to each loading condition are discussed separately. First, consider the effect of a unit force in the ith direction acting at the point $\vec{\xi}$ and beginning at time zero. Eq. (26) becomes

$$\dot{\hat{s}} - A\hat{s}'' = B_i \delta'(\alpha) H(t). \tag{28}$$

Its solution is well known (e.g. Carslaw and Jaeger, 1959)

$$\hat{s}(\alpha, t) = -\frac{B_i}{2A} \operatorname{erf}\left(\frac{\alpha}{2\sqrt{At}}\right) \tag{29a}$$

where erf is the error function (Abramowitz and Stegun, 1964) and defined in the following form:

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp\left(-\frac{z^2}{2}\right) \mathrm{d}z. \tag{29b}$$

Substituting $\hat{s}$ in Eq. (29a) into Eqs. (20a), (20b) and (23) leads to the second-order derivative for displacements, electric potential and temperature in the transform domain as

$$\hat{u}_{ij}'' = \frac{B_i c_j}{4A^{3/2}\sqrt{t}} \exp\left(-\frac{\alpha^2}{4At}\right) - h_{ji}\delta(\alpha)H(t)$$

$$\hat{\phi}_i'' = \frac{B_i f}{4A^{3/2}\sqrt{t}} \exp\left(-\frac{\alpha^2}{4At}\right) - q_i\delta(\alpha)H(t)$$

$$\hat{\theta}_i'' = \frac{B_i g \alpha}{8A^{5/2}\sqrt{t^3}} \exp\left(-\frac{\alpha^2}{4At}\right) + v_i\delta'(\alpha)H(t) \tag{30}$$

where

$$f = \frac{1}{a}\left[\left(e_{mnk} + \frac{T_0 \lambda_{kn} \pi_m}{\rho c_\varepsilon}\right) z_m z_n c_k + \frac{T_0 \pi_m z_m}{\rho c_\varepsilon}\right]$$

$$q_i = \frac{1}{a}\left(e_{mnk} + \frac{T_0 \lambda_{kn} \pi_m}{\rho c_\varepsilon}\right) z_m z_n h_{ki}$$

$$v_i = \frac{T_0}{c_\varepsilon}(\lambda_{mn} z_n h_{mi} - \pi_k z_k q_i). \tag{31}$$

Next, the infinite space response to a unit charge acting at $\vec{\xi}$ beginning at time zero is presented. Since such derivation presents no elements of novelty, we dispense with details and merely list the results:

$$\hat{u}''_{4i} = \frac{Cc_i}{4A^{3/2}\sqrt{t}}\exp\left(-\frac{\alpha^2}{4At}\right) - d_i\delta(\alpha)H(t)$$

$$\hat{\phi}''_4 = \frac{Ce}{4A^{3/2}\sqrt{t}}\exp\left(-\frac{\alpha^2}{4At}\right) - q\delta(\alpha)H(t)$$

$$\hat{\theta}''_4 = \frac{\alpha g C}{8A^{5/2}\sqrt{t^3}}\exp\left(-\frac{\alpha^2}{4At}\right) + w\delta'(\alpha)H(t). \tag{32}$$

Finally, the infinite space response to a unit pulse heat source acting at time zero, at the point $\vec{\xi}$ within three-dimensional solid is discussed in a similar way. By solving the following equation

$$\dot{\hat{s}} - A\hat{s}'' = \frac{1}{T_0}\delta(\alpha)\delta(t), \tag{33}$$

we arrive at (Carslaw and Jaeger, 1959)

$$\hat{s}(\alpha, t) = \frac{1}{T_0\sqrt{4\pi At}}\exp\left(-\frac{\alpha^2}{4At}\right). \tag{34}$$

As a result, the derivatives of Green's function components in transform domain are expressed in the following forms

$$\hat{u}''_{5i} = \frac{c_i\alpha}{4\pi T_0(At)^{3/2}}\exp\left(-\frac{\alpha^2}{4At}\right)$$

$$\hat{\phi}''5 = \frac{\alpha e}{4\sqrt{\pi}T_0(At)^{3/2}}\exp\left(-\frac{\alpha^2}{4At}\right)$$

$$\hat{\theta}''_5 = \frac{\alpha^2 g}{8\sqrt{\pi}T_0 A^{5/2} t^{3/2}}\exp\left(-\frac{\alpha^2}{4At}\right) - \frac{g}{4\sqrt{\pi}T_0(At)^{3/2}}\exp\left(-\frac{\alpha^2}{4At}\right). \tag{35}$$

It should be noted in comparison with thermoelasticity (Rice and Cleary, 1976; Rudnicki, 1987) that the terms with exponential function carry the solution from isentropic behavior at very short time to a final steady state form at very long period. The behavior will be easily demonstrated in the explicit form for the case of isotropic dielectric solid. In order to find u_{ij}, ϕ_i and θ_i, it is necessary to apply the inverse Radon transform defined in Eq. (18b) to Eqs. (30), (32) and (35). As a result, the responses to the above three loading conditions are listed as

$$u_{ij}\left(\vec{x}-\vec{\xi}\right)=-\frac{1}{8\pi^2}\oint_{|z|=1}\frac{B_i(\vec{z})c_j(\vec{z})}{4\sqrt{t}A^{3/2}(\vec{z})}\exp\left(-\frac{\left[\vec{z}\cdot\left(\vec{x}-\vec{\xi}\right)\right]^2}{4A(\vec{z})t}\right)\mathrm{d}s(\vec{z})+\frac{H(t)}{8\pi^2 r}\int_0^{2\pi}h_{ji}\,\mathrm{d}\psi$$

$$\phi_i\left(\vec{x}-\vec{\xi}\right)=-\frac{1}{8\pi^2}\oint_{|z|=1}\frac{fB_i(\vec{z})}{4\sqrt{t}A^{3/2}(\vec{z})}\exp\left(-\frac{\left[\vec{z}\cdot\left(\vec{x}-\vec{\xi}\right)\right]^2}{4A(\vec{z})t}\right)\mathrm{d}s(\vec{z})+\frac{H(t)}{8\pi^2 r}\int_0^{2\pi}q_i\,\mathrm{d}\psi$$

$$\theta_i\left(\vec{x}-\vec{\xi}\right)=-\frac{1}{8\pi^2}\oint_{|z|=1}\frac{\alpha gB_i(\vec{z})}{8\sqrt{t^3}A^{5/2}(\vec{z})}\exp\left(-\frac{\left[\vec{z}\cdot\left(\vec{x}-\vec{\xi}\right)\right]^2}{4A(\vec{z})t}\right)\mathrm{d}s(\vec{z})+\frac{H(t)}{8\pi^2 r^2}\int_0^{2\pi}\frac{\partial v_i}{\partial b}\,\mathrm{d}\psi$$

$$u_{4i}\left(\vec{x}-\vec{\xi}\right)=-\frac{1}{8\pi^2}\oint_{|z|=1}\frac{c_iC}{4\sqrt{t}A^{3/2}(\vec{z})}\exp\left(-\frac{\left[\vec{z}\cdot\left(\vec{x}-\vec{\xi}\right)\right]^2}{4A(\vec{z})t}\right)\mathrm{d}s(\vec{z})+\frac{H(t)}{8\pi^2 r}\int_0^{2\pi}d_i\,\mathrm{d}\psi$$

$$\phi_4\left(\vec{x}-\vec{\xi}\right)=-\frac{1}{8\pi^2}\oint_{|z|=1}\frac{eC}{4\sqrt{t}A^{3/2}(\vec{z})}\exp\left(-\frac{\left[\vec{z}\cdot\left(\vec{x}-\vec{\xi}\right)\right]^2}{4A(\vec{z})t}\right)\mathrm{d}s(\vec{z})+\frac{H(t)}{8\pi^2 r}\int_0^{2\pi}q\,\mathrm{d}\psi$$

$$\theta_4\left(\vec{x}-\vec{\xi}\right)=-\frac{1}{8\pi^2}\oint_{|z|=1}\frac{\alpha gC}{4\sqrt{t}A^{5/2}(\vec{z})}\exp\left(-\frac{\left[\vec{z}\cdot\left(\vec{x}-\vec{\xi}\right)\right]^2}{4A(\vec{z})t}\right)\mathrm{d}s(\vec{z})+\frac{H(t)}{8\pi^2 r^2}\int_0^{2\pi}\frac{\partial w}{\partial b}\,\mathrm{d}\psi$$

$$u_{5i}\left(\vec{x}-\vec{\xi}\right)=-\frac{1}{8\pi^2}\oint_{|z|=1}\frac{C_i\vec{z}\cdot\left(\vec{x}-\vec{\xi}\right)}{4\sqrt{\pi}\sqrt{t^3}A^{3/2}(\vec{z})}\exp\left(-\frac{\left[\vec{z}\cdot\left(\vec{x}-\vec{\xi}\right)\right]^2}{4A(\vec{z})t}\right)\mathrm{d}s(\vec{z})$$

$$\phi_5\left(\vec{x}-\vec{\xi}\right)=-\frac{1}{8\pi^2}\oint_{|z|=1}\frac{e\vec{z}\cdot\left(\vec{x}-\vec{\xi}\right)}{4\sqrt{\pi}\sqrt{t^3}A^{3/2}(\vec{z})}\exp\left(-\frac{\left[\vec{z}\cdot\left(\vec{x}-\vec{\xi}\right)\right]^2}{4A(\vec{z})t}\right)\mathrm{d}s(\vec{z})$$

$$\theta_5\left(\vec{x}-\vec{\xi}\right)=-\frac{1}{8\pi^2}\oint_{|z|=1}\frac{\left[\vec{z}\cdot\left(\vec{x}-\vec{\xi}\right)\right]^2 g-2gAt}{8\sqrt{\pi}t^{5/2}A^{5/2}(\vec{z})}\exp\left(-\frac{\left[\vec{z}\cdot\left(\vec{x}-\vec{\xi}\right)\right]^2}{4A(\vec{z})t}\right)\mathrm{d}s(\vec{z}) \tag{36}$$

where $r=|\vec{x}-\vec{\xi}|$, $\vec{z}\cdot(\vec{x}-\vec{\xi})=rb$, and b and ψ are shown in Fig. 1. During the derivation, the following identity was used

$$\oint\delta'(x-x')f(x)\,\mathrm{d}x=-\oint\delta(x-x')\frac{\partial f(x)}{\partial(x)}\,\mathrm{d}x. \tag{37}$$

It is noted in Eq. (36) that the exponential terms carry the transient behavior at very short time and steady state at very long time, while the integrals defined by the circle ψ have similar behavior to the

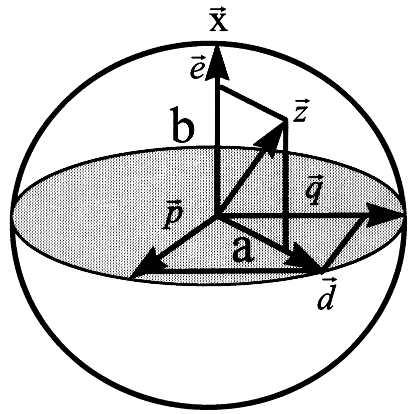

Fig. 1. Illustration of variables used in the evaluation of surface integrals.

static case and depends on the orientation of vector $\vec{x}-\vec{\xi}$ and not on the magnitude of $1/|\vec{x}-\vec{\xi}|$ (Deeg, 1980). Hence, it should not cause any numerical problem.

5. Reduced Green's function for isotropic dielectric solid

As an illustration of the results established so far, also as a check of the solutions, the details are worked out for an isotropic dielectric solid where the piezoelectricity and pyroelectricity vanish. As a consequence, the material constant tensors are simplified as

$$c_{ijkl}=\lambda\delta_{ij}\delta_{kl}+\mu\big(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}\big),\quad k_{ij}=k\delta_{ij},\quad \varepsilon_{ij}=\varepsilon\delta_{ij}$$

$$\lambda_{ij}=(3\lambda+2\mu)\alpha_0\delta_{ij}$$

$$e_{ijk}=0,\quad \pi_i=0 \tag{38}$$

where λ and μ are the Lame's constants, k is the thermal conductivity, α_0 is the coefficient of thermal expansion, ε is the dielectric constant. It is assumed that strain and electric field are unable to produce entropy, that is, $s=(\rho c_\varepsilon\theta)/T_0$. Hence,

$$A=\frac{kz_iz_i}{\rho c_\varepsilon},\quad B_i=C=0. \tag{39}$$

Eqs. (20a), (20b) and (23) become

$$\hat{u}_k''=-c_k\hat{s}'-h_{ki}F_i\delta(\alpha)H(t), \tag{40a}$$

$$\hat{\phi}_4''=-q\Phi_e\delta(\alpha)H(t) \tag{40b}$$

$$\hat{\theta}_5''=\frac{T_0\hat{s}''}{\rho c_\varepsilon} \tag{40c}$$

where

$$h_{ki} = \frac{1}{\mu z_m z_m}\left[\delta_{ki} - \frac{z_i z_k}{2(1-\nu)z_m z_m}\right]$$

$$c_k = -\frac{(3\lambda+2\mu)\alpha_0}{\rho c_\varepsilon \mu z_m z_m}\left[\delta_{ki} - \frac{z_i z_k}{2(1-\nu)z_m z_m}\right]$$

$$q = \frac{1}{\varepsilon z_m z_m} \tag{41}$$

where ν is the Poisson's ratio and $\nu = \lambda(1-2\mu)/2\mu$. Applying the inverse Radon transform to Eq. (40b) yields

$$\phi_4\left(\vec{x}-\vec{\xi}\right) = \frac{1}{8\pi^2}\oint_{|z|=1}\frac{1}{\varepsilon z_m z_m}\delta(\alpha)H(t)\,\mathrm{d}s(\vec{z}). \tag{42}$$

Using the result given in Appendix A (Eq. (A10)), we obtain

$$\phi_4\left(\vec{x}-\vec{\xi}\right) = \frac{1}{4\pi\varepsilon r}H(t) \tag{43}$$

where $r = |\vec{x}-\vec{\xi}|$. If ϕ_4 is normalized by ε, the solution is exactly the same as that in the three-dimensional potential problem (Courant and Hilbert, 1963). From Eqs. (40a), (40b) and (41), we obtain

$$\hat{\theta}_5'' = \frac{1}{\rho c_\varepsilon}\frac{1}{\sqrt{4\pi A t}}\frac{1}{4\pi^2 t^2}(\alpha^2 - 2At)\exp\left(-\frac{\alpha^2}{4At}\right)$$

$$\hat{u}_{5k}'' = \frac{(3\lambda+2\mu)\alpha_0}{\rho c_\varepsilon \mu z_m z_m (2At)\sqrt{4\pi A t}}\alpha z_i\left[\delta_{ki} - \frac{z_i z_k}{2(1-\nu)z_m z_m}\right]\exp\left(-\frac{\alpha^2}{4At}\right)$$

$$\hat{u}_{ij}'' = \frac{1}{(2At)\sqrt{4\pi A t}}\alpha\left[\delta_{ij} - \frac{z_i z_j}{2(1-\nu)z_m z_m}\right]\exp\left(-\frac{\alpha^2}{4At}\right). \tag{44}$$

Taking the inverse Radon transform and resorting to the results from Eqs. (A11) to (A13) yield

$$\theta_5 = \frac{1}{8\pi r k}\frac{\bar{\eta}}{\sqrt{\pi}t}\exp\left(-\frac{\bar{\eta}^2}{4}\right)$$

$$u_{5i} = \frac{1}{4\pi}\frac{(3\lambda+2\mu)\alpha_0}{k(\lambda+2\mu)}\left\{\frac{x_i}{r}\frac{1}{\bar{\eta}^2 t}\left[\frac{\bar{\eta}^2}{\sqrt{\pi}}\exp\left(-\frac{\bar{\eta}^2}{4}\right) - \mathrm{erf}\left(\frac{\bar{\eta}}{2}\right)\right]\right\}$$

$$u_{ij} = \frac{1}{16\pi r}\frac{1}{\mu(1-\nu)}\left[\frac{x_i}{r}\frac{x_j}{r} + (3-4\mu)\delta_{ij}\right]H(t) \tag{45}$$

where $i,j = 1,2,3$ and where $\bar{\eta} = r/\sqrt{kt/\rho c_\varepsilon}$. The other components vanish due to the lack of full coupling. It is observed that the solutions are exactly the same as those given by Dargush and Banerjee (1990).

6. Conclusions

The "displacement–electric potential–temperature" type boundary integral equation for general three-dimensional thermo-piezoelastic problem is presented. The corresponding Green's solutions are obtained in integral form. The method may prove to be quite attractive primarily for two reasons. First, no volume discretization is required. As a result, significant savings in modeling effort are possible, particularly for a bulky body. The second attractive feature involves the ability of the boundary-only approach to capture the steep thermal gradients that are often associated with severe transients. The reduction to the isotropic dielectric case shows the solution's consistency with the existing ones.

Acknowledgements

The author is grateful to Prof. C.T. Sun in Purdue University for his comments and anonymous reviewers for their constructive suggestions.

Appendix A

The evaluation of the inverse Radon transform greatly depends on the manipulation of the surface integral over a unit sphere. By using the algorithm illustrated by Deeg (1980) and Wang and Achenbach (1993), the process to evaluate the surface integral

$$f(\vec{x}) = -\frac{1}{8\pi^2}\int_{|\vec{z}|=1} \hat{f}(\alpha, \vec{z})\,\mathrm{d}s(\vec{z}) \tag{A1}$$

is outlined below.

As shown in Fig. 1, let $\vec{e}$ be a unit vector in the direction of $\vec{x}$. Thus,

$$\vec{x} = r\vec{e}, \tag{A2a}$$

or

$$e_i = \frac{x_i}{r}, \tag{A2b}$$

$$r = |\vec{x}|. \tag{A2c}$$

$\vec{d}$ is a unit vector and

$$\vec{d}\cdot\vec{e} = 0. \tag{A3}$$

In the $\vec{d}-\vec{e}$ plane, $\vec{z}$ is decomposed into

$$\vec{z} = a\vec{d} + b\vec{e} \tag{A4}$$

where

$$\sqrt{a^2+b^2}=1. \tag{A5}$$

As a result, $\alpha=\vec{z}\cdot\vec{x}=rb$. It is noted that b ranges from -1 reflecting the negative $\vec{x}$ direction to 1. Furthermore, $\vec{d}$ may be expressed in terms of another parameter ψ. On the plane normal to $\vec{e}$,

$$\vec{d}=\cos\psi\vec{p}+\sin\psi\vec{q} \tag{A6}$$

where $\vec{p}$, $\vec{q}$ and $\vec{e}$ form a right-hand coordinate. Hence,

$$\vec{z}=\frac{r\sqrt{1-b^2}}{\sqrt{x_1^2+x_2^2}}\left(\frac{x_2}{r}\cos\psi-\frac{x_1}{r}\frac{x_3}{r}\sin\psi,\ -\frac{x_1}{r}\cos\psi-\frac{x_2}{r}\frac{x_3}{r}\sin\psi,\ \frac{x_1^2+x_2^2}{r^2}\sin\psi\right)+b\left(\frac{x_1}{r},\frac{x_2}{r},\frac{x_3}{r}\right). \tag{A7}$$

As a result, the surface integral can be expressed in terms of b and ψ as follows

$$f(\vec{x})=-\frac{1}{8\pi^2}\int_0^{2\pi}\int_{-1}^{1}\bar{f}\big(rb,\vec{z}(b,\psi)\big)\,\mathrm{d}b\mathrm{d}\psi. \tag{A8}$$

Using the property of Dirac function, $\delta(\alpha)=\delta(rb)=(1/r)\delta(b)$, we have

$$f(\vec{x})=-\frac{1}{8\pi^2}\int_{|\vec{z}|=1}\hat{f}\big(\alpha,\vec{z}\big)\delta(\alpha)\,\mathrm{d}s(\vec{z})=-\frac{1}{8\pi^2 r}\int_0^{2\pi}\bar{f}(0,\vec{z}(0,\psi))\,\mathrm{d}\psi. \tag{A9}$$

Towards the end, the integrals used in the section for isotropic dielectricity are derived as follows

(1). $$\oint_{|\vec{z}|=1}\delta(\alpha)\,\mathrm{d}s(z)=\int_0^{2\pi}\int_{-1}^{1}\delta(rb)\,\mathrm{d}b\mathrm{d}\psi=\frac{2\pi}{r}$$

(2). $$\oint_{|\vec{z}|=1}(\alpha^2-2\bar{A}t)\exp\left(-\frac{\alpha^2}{4At}\right)\mathrm{d}s(z)=\int_0^{2\pi}\int_{-1}^{1}(r^2b^2-2\bar{A}t)\exp\left(-\frac{\alpha^2}{4\bar{A}t}\right)\mathrm{d}b\mathrm{d}\psi \tag{A10}$$

where $\bar{A}=k/(\rho c_\varepsilon)$. It is easily shown by using integration by parts that the above equation is

$$=-\frac{8\pi(\bar{A}t)^{3/2}}{r}\bar{\eta}\exp\left(-\frac{\bar{\eta}^2}{4}\right) \tag{A11}$$

where $\bar{\eta}=r/\sqrt{\bar{A}t}=r/\sqrt{kt/\rho c_\varepsilon}$.

(3).

$$\oint_{|\bar{z}|=1} \alpha z_i\left(\delta_{ki} - \frac{z_i z_k}{2(1-\nu)}\right)\exp\left(-\frac{\alpha^2}{4\bar{A}t}\right)\mathrm{d}s(\bar{z}) = \frac{2\pi(1-2\nu)x_k}{(1-\nu)}r\int_0^1 b^2\exp\left(-\frac{(rb)^2}{4\bar{A}t}\right)\mathrm{d}b$$
$$= \frac{2\pi(1-2\nu)x_k}{(1-\nu)r}r\left(-\frac{2\bar{A}t}{r^2}\right)\left[\exp\left(-\frac{\bar{\eta}^2}{4}\right) - \frac{\sqrt{\pi}}{\bar{\eta}}\mathrm{erf}\left(\frac{\eta}{2}\right)\right] \quad \text{(A12)}$$
$$= -\frac{2\pi(1-2\nu)\bar{\eta}}{(1-\nu)}\frac{x_k}{r}\left[\exp\left(-\frac{\bar{\eta}^2}{4}\right) - \frac{\sqrt{\pi}}{\bar{\eta}}\mathrm{erf}\left(\frac{\eta}{2}\right)\right]$$

where $\mathrm{erf}(\bar{\eta}/2) = (2/\sqrt{\pi})\int_0^{\bar{\eta}/2}\exp(-z^2/2)\,\mathrm{d}z$.

(4).

$$\oint_{|\bar{z}|=1}\left(\delta_{ki} - \frac{z_i z_k}{2(1-\nu)}\right)\delta(\alpha)\exp\left(-\frac{\alpha^2}{4At}\right)\mathrm{d}s(z) = \frac{\pi}{r}\delta_{ki} - \frac{1}{2r(1-\nu)}\int_0^{2\pi} z_i z_k|_{b=0}\,\mathrm{d}\psi \quad \text{(A13)}$$
$$= \frac{\pi}{r}\delta_{ki} - \frac{\pi}{2r(1-\nu)}\left(\delta_{ki} - \frac{x_k}{r}\frac{x_i}{r}\right) = \frac{\pi}{2r(1-\nu)}\left[(3-4\nu)\delta_{ki} + \frac{x_k}{r}\frac{x_i}{r}\right].$$

References

Abramowitz, M., Stegun, I.A., 1964. Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables, 5th ed. Dover, New York.

Ashisa, F., Tauchert, T.R., Noda, N., 1983. Response of a piezothermoelastic plate of crystal class 6 mm subject to axisymmetric heating. Int. J. Engrg. Sci 31 (3), 373–384.

Bacon, D.J., Barnett, D.M., Scattergood, R.O., 1979. Anisotropic continuum theory of lattice defects. Progress in Materials Science 23, 51–263.

Boley, B.A., Weiner, J.H., 1960. Theory of Thermal Stresses. Wiley, New York.

Carslaw, H.S., Jaeger, J.C., 1959. Conduction of Heat in Solids, 2nd ed. Clarendon Press, Oxford.

Chen, T., 1993. Green functions and the non-uniform transformation problem in a piezoelectric medium. Mech. Res. Comm 20 (3), 271–278.

Chen, T., Lin, F., 1993. Numerical evaluation of derivatives of the anisotropic piezoelectric Green's functions. Mech. Res. Comm 20, 501–506.

Courant, R., Hilbert, D., 1962. Methods of Mathematical Physics, vol. II. Interscience, New York.

Dargush, G.F., Banerjee, P.K., 1990. Boundary element methods in three-dimensional thermoelasticity. Int. J. Solids and Structures 26 (2), 199–216.

Deans, S.R., 1983. The Randon Transform and Some of its Applications. Wiley–Interscience, New York.

Deeg, W.F.I., 1980. The analysis of dislocation, crack and inclusion problems in a piezoelectric medium. Ph.D Dissertation, Stanford University.

Ding, H.J., Chen, B., Lian, G., 1997. On the Green's functions of 2-phase transversely isotropic piezoelectric media. Int. J. Solids and Structures 34 (23), 3041–3057.

Dunn, M.L., 1993. Exact relation between the thermoelectroelastic moduli of heterogeneous materials. Proc. R. Soc. London, A 441, 549–557.

Dunn, M.L., 1994a. Electroelastic Green's functions for transversely isotropic piezoelectric media and their application to the solution of inclusion and inhomogeneity problems. Int. J. Engng. Sci 32 (1), 119–131.

Dunn, M.L., 1994b. Micromechanics of coupled electroelastic composites: effective thermal expansion and pyroelectric coefficients. J. Apply. Physics 73 (10), 5131–5140.

Gelfand, I.M., Shilow, G.E., 1964. Generalized Functions, vol. 1. Academic Press, New York.

Jiang, L.Z., Lee, J.S., 1994. Plane orthotropic layer by transfer matrix-spline boundary element. ASCE, J. Engineering Mechanics 120 (5), 1026–1040.

Jiang, L.Z., Sun, C.T., 1999. Crack growth behavior in piezoelectrics under cyclic loads. Ferroelectrics, in press.

Jonnalagadda, K.D., Blandford, G.E., Tauchert, T.R., 1994. Piezothermoelastic composite plate analysis using first order shear deformation theory. Computers and Structures 51 (1), 79–89.

Khutoryansky, N.M., Sosa, H.A., 1995. Dynamical representation formulas and fundamental solutions for piezoelectricity. Int. J. Solids and Structures 33 (22), 3307–3325.

Lee, J.S., Jiang, L.Z., 1994a. A boundary integral formulation and 2D fundamental solution for piezoelectric media. Mech. Res. Comm 21, 47–54.

Lee, J.S., Jiang, L.Z., 1994b. Thermal stresses in piezoelectric materials: a boundary integral formulation. In: Proceedings of SPACE 94, Albuquerque, New Mexico, vol. 1, pp. 336–347.

Lee, J.S., Jiang, L.Z., 1996. Exact electroelastic analysis of piezoelectric laminae via state space approach. Int. J. Solids Structures 7 (33), 1996.

Massalas, C.V., Kalpakidis, V.K., Foutsitzi, G., 1994. Some comments on the extended Tiersten's theory of thermoelectroelasticity. Mech. Res. Comm 21 (4), 343–351.

Mura, T., 1984. Micromechanics of Defects in Solids, 2nd ed. Martinus Nijhoff, Dordrecht.

Norris, A.N., 1994. Dynamic Green's functions in anisotropic piezoelectric, thermoelastic and poroelastic solids. Proc. R. Soc. London, A, 447, 175–189.

Nowacki, W., 1979. Foundation of linear piezoelectricity. In: Parkus, H. (Ed.), Electromagnetic Interaction in Elastic Solids. Springer–Verlag, Wien.

Nye, J.F., 1957. Physical Properties of Crystals: Their Representation by Tensor and Matrices. Clarendon Press, Oxford.

Pak, Y.E., 1992. Linear electro-elastic fracture mechanics of piezoelectric materials. Int. J. of Fract 54, 79–100.

Rice, J.R., Cleary, M.P., 1976. Some basic stress diffusion solutions for fluids saturated elastic porous media and compressible constituents. Rev. Geophys. Space Phys 14 (2), 227–241.

Rudnicki, J.W., 1987. Fluid mass source and point forces in linear elastic diffusive solids. Mech. Mater 5, 383–393.

Sosa, H., 1991. Plane problems in piezoelectric media with defects. Int. J. Solids and Structures 28 (4), 491–505.

Suo, Z., Kuo, C.M., Barnett, D.M., Willis, J.R., 1992. Fracture mechanics for piezoelectric ceramics. J. Mech. Phy. Solids 40 (4), 734–765.

Tiersten, H.F., 1971. On the non-linear equation of thermoelectroelasticity. J. Engng. Sci 9, 587–604.

Wang, C.Y., Achenbach, J.D., 1993. A new method to obtain 3D Green's functions for anisotropic solids. Wave Motion 18, 273–289.

Wang, C.Y., Achenbach, J.D., 1994. Three dimensional time-harmonic elastodynamic Green's functions for anisotropic solids. Proc. R. Soc. Lodon, A (1995) 449, 441–458.

Tzou, H.S. (Ed.), 1993. Intelligent Structures, Materials and Vibrations. ASME, DE, vol. 59. pp. 27–54.